Topological spaces

Def: If X is a set, a topology on X is a collection $\mathcal{T}$ of subsets of X (i.e. $\mathcal{T} \subseteq \mathcal{P}(X)$) s.t.

1.) $\emptyset, X \in \mathcal{T}$

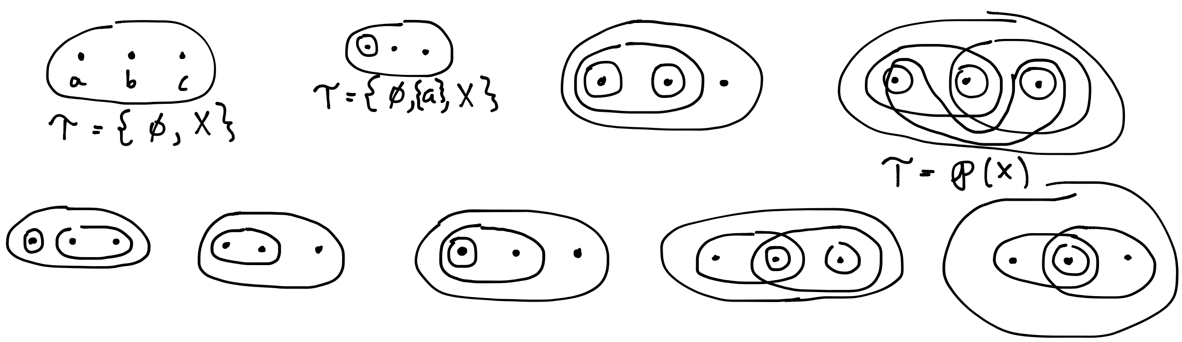
2.) If $\mathcal{T}' \subseteq \mathcal{T}$, then $\bigcup_{S \in \mathcal{T}'} S \in \mathcal{T}$.
(i.e. the union of subsets in $\mathcal{T}$ is also in $\mathcal{T}$)

3.) If $\mathcal{T}' \subseteq \mathcal{T}$ is a finite set of subsets, then
$\bigcap_{S \in \mathcal{T}'} S \in \mathcal{T}$. (finite intersections are in $\mathcal{T}$.)

If X is a topological space w/ topology $\mathcal{T}$, then $U \subseteq X$ is open if $U \in \mathcal{T}$.

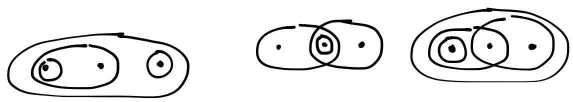
Ex: The standard topology on $\mathbb{R}$ was described in the last section. We already proved that it satisfies all the axioms and thus is indeed a topological space.

Ex: Let $X = \{a, b, c\}$. What are the possible topologies on X?



You can get the rest by permuting the elements.

Why aren't the following topologies?



Ex: If X is a set, then the power set $\mathcal{P}(X)$ is a topology on X. It's called the <u>discrete topology</u>. Every set is open.

Ex: Let $X = \mathbb{R}$, and let $\mathcal{T} = \{ S \subseteq \mathbb{R} \mid \mathbb{R} - S \text{ is finite or } S = \emptyset \}$

i.e. the open sets are the subsets of $\mathbb{R}$ w/ finite complements.

1.) $\emptyset \in \mathcal{T}$ and $\mathbb{R} - \mathbb{R} = \emptyset$, so $\mathbb{R} \in \mathcal{T}$

2.) If $S = \bigcup_{S' \in \mathcal{T}'} S'$ then $\mathbb{R} - S = \bigcap_{S' \in \mathcal{T}'} (\mathbb{R} - S')$, which is finite (or $\mathbb{R}$), since $\mathbb{R} - S'$ is finite (or $\mathbb{R}$).

3.) If $S = S_1 \cap S_2 \cap \ldots \cap S_n$, $S_i \in \mathcal{T}$, then $\mathbb{R} - S = \bigcup (\mathbb{R} - S_i)$, which is $\mathbb{R}$ or a finite union of finite sets and thus finite.

This is called the finite complement topology or the cofinite topology.

Note that we didn't actually use the fact that $X = \mathbb{R}$. In fact, you can put the cofinite topology on any set.

Ex: Let X be an infinite set, $\mathcal{T} = \{ S \subseteq X \mid S \text{ is finite or } S = X \}$

This is <u>not</u> a topology. For example, let $Y \subsetneq X$ be a proper infinite set. Then $Y = \bigcup_{y \in Y} \{y\}$, but $Y \notin \mathcal{T}$.